The partition function of a multi-component Coulomb gas on a circle

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Abstract

We study a two-dimensional Coulomb gas consisting of a mixture of particles carrying various positive multiple integer charges, confined on a unit circle. We consider the system in the canonical and grand canonical ensembles, and attempt to calculate the partition functions analytically, using Toeplitz and confluent Vandermonde determinants. Just like in the simple one-component system (Dyson gas), the partition functions simplify at special temperature $\beta = 2$, allowing us to find compact expressions for them.

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1. Introduction and summary

Classical two-dimensional Coulomb gas at finite temperature is a prototype example for learning about phase transitions. There are various interesting simplified versions of the system. One can confine the particles to move on a unit circle and yet obtain an important toy model for various analytical techniques, which also turns out to be related to other rather different physical systems [1]. Typically one restricts all charges to have the same strength (which can be normalized to unity, leaving the temperature as the only parameter) but possibly different signs. Such systems have been solved exactly in the bulk and on the circle in the whole stability range of temperatures [2–5][4]. In the simplest system (the Dyson gas [7]) all the charges are confined on the circle and have the same sign and strength. Even the Dyson gas at finite temperature is very interesting; among other things it gives a physical realization of different random matrix ensembles, just by adjusting the temperature [8]. The canonical partition function can be calculated analytically at generic temperature and has a simple form

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[4] Note also that a two-component system of charge ratio 1:2 was considered in [6].

[7, 9], but the grand canonical partition function is much more complicated [1–3]. However, at special inverse temperature $\beta = 2$ it simplifies drastically into a very simple form.

In this paper, we consider a slightly more general version of the system, consisting of a mixture of particles carrying different integer charges, albeit with the same (positive) sign[5]. This generalization makes the calculations much more complicated. At special inverse temperature $\beta = 2$ there are again simplifications, but now even the canonical partition function is more difficult to calculate. We develop techniques using confluent Vandermonde determinants, Toeplitz determinants and generating functional methods, to arrive at a compact expression for the partition function (at $\beta = 2$) for any combination of positive integer charges[6]. For the grand canonical partition function, we also find a compact expression, but it is more complicated as it still contains an infinite series for which we have not been able to find an explicit summation.

Our analysis is mostly restricted to the special inverse temperature $\beta = 2$ and to positive integer charges. It would be important to find generalizations of our techniques for all temperatures. This would also allow a treatment of the case where some of the particles carry negative charges [3–5, 10]. Another restriction of the present analysis is the number of particles. Although our formulae are exact, they can in practice only be used for small numbers of multiply-charged particles due to the rapidly increasing complexity of the expressions. It is perhaps possible to overcome this restriction; instead of exact formulae one may seek asymptotic approximations for a large number of particles, e.g. with the help of asymptotics of large Toeplitz determinants[7]. Together with a generalization to arbitrary temperatures, this might shed light on the thermodynamic limit of these systems, including phase transitions and analytic properties of the partition functions.

The paper is organized as follows. We introduce the multi-component Coulomb gas in the following section. In section 3, we construct the system from the one-component Dyson gas by placing several unit charges at the same point and removing the infinite self-energy, and present a result for the canonical partition function of the multi-component gas. Technical details of the calculation involving manipulations of confluent Vandermonde determinants are postponed to appendix A. In section 4 we develop the former construction into a generating functional method to compute the canonical and grand canonical partition functions. We end by giving an explicit example of applying our methods in section 5.

2. Setup

The Hamiltonian of the standard Dyson gas with N particles is given by

$$H_{\mathrm{D}} = \sum_{1 \leqslant i < j}^{N} V(t_i, t_j), \tag{1}$$

where

$$V(t_i, t_j) = -\log |e^{it_i} - e^{it_j}| \tag{2}$$

is the two-dimensional Coulombic potential between the particles i and j which are located at e^{it_i} and at e^{it_j} on the unit circle, respectively. The definition can be naturally extended for particles having multiple positive integer charges. Let us take N_1 particles of charge +1, N_2

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[5] Equivalently, one could consider a mixture of Dyson gases, at different (initial) temperatures.

[6] An explicit analytic form can then be found for any given charge expression with the help of a straightforward computer algorithm.

[7] We will discuss the behavior of the partition function in the limit of a large number of +1 charges, in a different framework (decaying branes in string theory) in a future publication [11].

particles of charge $+2, \ldots, N_{n_{\max}}$ particles of charge $+n_{\max}$, labeled by positions $\{t_1^{(1)}, \ldots, t_{N_1}^{(1)}\}$, $\{t_1^{(2)}, \ldots, t_{N_2}^{(2)}\}, \ldots, \{t_1^{(n_{\max})}, \ldots, t_{N_{n_{\max}}}^{(n_{\max})}\}$, respectively. They interact via

$$V_{nm}\big(t_i^{(n)}, t_j^{(m)}\big) = -nm \log \big| e^{it_i^{(n)}} - e^{it_j^{(m)}} \big|. \tag{3}$$

The Hamiltonian of this system is then given by

$$
\begin{aligned}
H &= \sum_{\text{pairs}} V_{nm}\big(t_i^{(n)}, t_j^{(m)}\big) \\
&= -\sum_{n=1}^{n_{\max}} n^2 \sum_{1 \leqslant i < j}^{N_n} \log \big| e^{it_i^{(n)}} - e^{it_j^{(n)}} \big| - \sum_{1 \leqslant n < m}^{n_{\max}} nm \sum_{i=1}^{N_n} \sum_{j=1}^{N_m} \log \big| e^{it_i^{(n)}} - e^{it_j^{(m)}} \big|.
\end{aligned}
\tag{4}
$$

The canonical partition function is defined by

$$Z_{\mathrm{C}}(N_1, N_2, \ldots, N_{n_{\max}}) = \frac{1}{\prod_n N_n!} \int \left[\prod_{n=1}^{n_{\max}} \prod_{i=1}^{N_n} \frac{\mathrm{d}t_i^{(n)}}{2\pi} \right] e^{-\beta H}. \tag{5}$$

In this paper we analyze Z_{C} at a fixed inverse temperature $\beta = 2$. The results are also used to calculate the grand canonical partition function[8]

$$Z_{\mathrm{G}}(\hat{z}_1, \ldots, \hat{z}_{n_{\max}}) = \sum_{N_1=0}^{\infty} (\hat{z}_1)^{N_1} \cdots \sum_{N_{n_{\max}}=0}^{\infty} (\hat{z}_{n_{\max}})^{N_{n_{\max}}} Z_{\mathrm{C}}(N_1, \ldots, N_{n_{\max}}), \tag{6}$$

where $\hat{z}_i$ are the fugacities which correspond to the different charges. We will start by a rather direct evaluation of the integrals over t_i in (5), and then go on to study a more advanced formulation involving generating functionals.

3. Z_{C} and confluent Vandermonde determinants

Let us study the partition function (5) at the particular inverse temperature $\beta = 2$. We denote $z_i = \exp(it_i)$. The partition function of the standard Dyson gas with the Hamiltonian (1) becomes

$$Z_{\mathrm{D}}(N) = \frac{1}{N!} \int \prod_{i=1}^{N} \frac{\mathrm{d}t_i}{2\pi} \prod_{1 \leqslant i < j}^{N} | e^{it_i} - e^{it_j} |^2 = \frac{1}{N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{2\pi i z_i} |\Delta(z_1, \ldots, z_N)|^2. \tag{7}$$

Here the integrand $\rho_{\mathrm{D}} = \exp(-2H_{\mathrm{D}})$ is the absolute value squared of the Vandermonde determinant

$$
\begin{aligned}
|\Delta(z_1, \ldots, z_N)|^2 &= \prod_{1 \leqslant i < j \leqslant N} |z_i - z_j|^2 = \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_N} z_1^{i_1 - 1} \cdots z_N^{i_N - 1} \right|^2 \\
&= \left| \sum_{\Pi} (-1)^{\Pi} \prod_{i=1}^{N} z_i^{\Pi(i)-1} \right|^2,
\end{aligned}
\tag{8}
$$

where Π denotes permutations of $1, 2, \ldots, N$ and $(-1)^{\Pi}$ is the sign of the permutation Π. Using $|\Delta|^2 = \Delta \Delta^*$ we can express the integrand as an analytic function of z_i,

$$|\Delta(z_1, \ldots, z_N)|^2 = \sum_{\Pi_1, \Pi_2} (-1)^{\Pi_1} (-1)^{\Pi_2} \prod_{i=1}^{N} z_i^{\Pi_1(i)-\Pi_2(i)}, \tag{9}$$

and doing the integrals in (7) simply picks the constant term in (9) such that $Z_{\mathrm{D}}(N) = 1$.

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[8] For an attempt to calculate the grand canonical partition function of this system, but in a different framework, see [12].
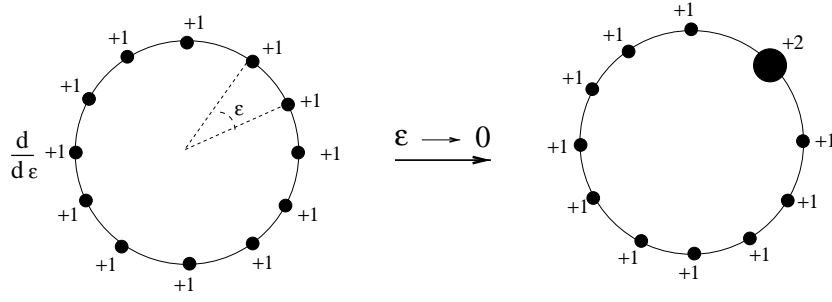
Figure 1. (The square root of) the Boltzmann weight of a state with one doubly-charged particle is obtained by forcing two unit charges to coincide. The infinite self-energy of the formed pair is removed by a derivative w.r.t. the angle ϵ between the particles.

The above analysis generalizes to the multi-component Coulomb gas. In fact,

$$\prod_{\text{pairs}} \left(z_i^{(n)} - z_j^{(m)} \right)^{nm} \tag{10}$$

can be expressed as a determinant of a confluent Vandermonde matrix (see, e.g. [13]). We present here some details of a simple proof since in the case of the Dyson gas it has an intuitive physical interpretation. The general treatment can be found in the appendix A.

The idea of the derivation goes as follows. We start from a single-component Dyson gas with at a lot of particles. We then form multiply-charged particles by placing several unit charges at the same point and removing the infinite self-energy of the formed particle. Let us start from the simplest case of only one double charge to see how this works (see also figure 1). We take a system with $N+2$ unit charges and denote $\tau_i = t_{N+i}$, $w_i = z_{N+i}$. Setting $\tau_2 = \tau_1 + \epsilon$ we immediately get

$$|\Delta(z_1, \ldots, z_{N+2})| = \prod_{1 \leqslant i < j \leqslant N} |z_i - z_j| \prod_{i=1}^{N} |z_i - w_1||z_i - w_2||w_1 - w_2|$$

$$= \prod_{1 \leqslant i < j \leqslant N} |z_i - z_j| \prod_{i=1}^{N} |z_i - w_1|^2 \epsilon + \mathcal{O}(\epsilon^2)$$

$$\equiv [\rho_2(z_1, \ldots, z_N, w_1)]^{1/2} \epsilon + \mathcal{O}(\epsilon^2), \tag{11}$$

where the leading result gives exactly the wanted Boltzmann weight ρ_2 times the exponential of the self-energy that we need to divide away. Notably, as depicted in figure 1 the above procedure equals differentiation,

$$[\rho_2(z_1, \ldots, z_N, w_1)]^{1/2} = \frac{1}{\epsilon} |\Delta(z_1, \ldots, z_{N+2})| + \mathcal{O}(\epsilon)$$

$$= \left| \frac{\Delta(z_1, \ldots, z_N, w_1, w_2)}{w_2 - w_1} \right| + \mathcal{O}(\epsilon)$$

$$\xrightarrow[\epsilon \to 0]{} \left| \frac{\partial}{\partial w_2} \Delta(z_1, \ldots, z_N, w_1, w_2) \right|_{w_2 = w_1}$$

$$= \left| \sum_{\Pi} (-1)^{\Pi} \prod_{i=1}^{N} z_i^{\Pi(i)-1} w_1^{\Pi(N+1)-1} [\Pi(N+2)-1] w_1^{\Pi(N+2)-2} \right|, \tag{12}$$

where (8) was used in the last step. In particular, from (12) one sees that ρ can be expressed as a determinant of a certain confluent Vandermonde matrix,

$$
A = \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 0 \\
z_1 & z_2 & & z_N & w_1 & 1 \\
z_1^2 & z_2^2 & & z_N^2 & w_1^2 & 2w_1 \\
\vdots & & & & & \vdots \\
z_1^{N+1} & z_2^{N+1} & \cdots & z_N^{N+1} & w_1^{N+1} & (N+1)w_1^N
\end{pmatrix}
\tag{13}
$$

which is obtained from the standard Vandermonde matrix by differentiation. A lengthy calculation gives the formula

$$
\begin{aligned}
Z_{\mathrm{C}}(N,1) &= \frac{1}{N!1!} \int \prod_{i=1}^{N} \frac{\mathrm{d}t_i}{2\pi} \frac{\mathrm{d}\tau}{2\pi} \rho_2(t_1,\ldots,t_N,\tau) \\
&= \frac{1}{N!} \oint \prod_{i=1}^{N} \frac{\mathrm{d}z_i}{2\pi i z_i} \frac{\mathrm{d}w_1}{2\pi i w_1} \sum_{\Pi_1,\Pi_2} (-1)^{\Pi_1}(-1)^{\Pi_2} \prod_{i=1}^{N} z_i^{\Pi_1(i)-\Pi_2(i)} \\
&\quad \times [\Pi_1(N+2)-1][\Pi_2(N+2)-1]w_1^{\Pi_1(N+1)+\Pi_1(N+2)-\Pi_2(N+1)-\Pi_2(N+2)} \\
&= \frac{1}{12}(N+3)(N+2)^2(N+1) = \binom{N+4}{4} + \binom{N+3}{4},
\end{aligned}
\tag{14}
$$

where the complex integrals again pick the constant term of the integrand. The result (14) equals the one obtained by using the Selberg integral [12].

We present the straightforward, but technical generalization to any number of charges in the appendix A. In particular, particles carrying higher charges ($n > 2$) are obtained by taking higher derivatives. The final result for Z_{C} in a general configuration is given in (A.14). Note that the result can be generalized to some special higher values of β by adjusting the charge configuration of the particles. This is possible if $\sqrt{\beta/2}q_i$ are positive integers for all the charges q_i of the system. For example, Z_{C} for a system with $\beta = 8$ and for $\tilde{N}_n$ particles of the charge n (with $n = 1,\ldots,n_{\max}$) is obtained by setting

$$
N_1 = N_3 = \cdots = 0; \qquad N_{2n} = \tilde{N}_n
\tag{15}
$$

in (A.14).

Similar results for Z_{C} as those of the appendix A can also be derived by using a more elegant method that involves a generating functional, which we will discuss next.

4. Generating functional method

Let us define a generating functional (see the appendix B for an alternative definition of $I[j]$),

$$
I[j] = \sum_{M=0}^{\infty} \frac{1}{M!} \int \left[\prod_{i=1}^{M} \frac{\mathrm{d}t_i}{2\pi} j(t_i) \right] \prod_{1 \leqslant i < j}^{M} |\,\mathrm{e}^{it_i} - \mathrm{e}^{it_j}|^{\beta}.
\tag{16}
$$

The weight of the standard Dyson gas ρ_{D} is then found by taking functional derivatives of $I[j]$,

$$
\rho_{\mathrm{D}}(t_1,\ldots,t_N) = \left. \frac{\delta^N I[j]}{\delta j(t_1)\cdots\delta j(t_N)} \right|_{j=0}.
\tag{17}
$$

For the special inverse temperature $\beta = 2$ we can use (9) to write (see, e.g. [14])

$$
\begin{aligned}
I[j] &= \sum_{M=0}^{\infty} \frac{1}{M!} \int \left[\prod_{i=1}^{M} \frac{\mathrm{d}t_i}{2\pi} j(t_i) \right] \sum_{\Pi_1, \Pi_2} (-1)^{\Pi_1} (-1)^{\Pi_2} \prod_{i=1}^{M} \mathrm{e}^{it_i [\Pi_1(i) - \Pi_2(i)]} \\
&= \sum_{M=0}^{\infty} \frac{1}{M!} \sum_{\Pi_1, \Pi_2} (-1)^{\Pi_1} (-1)^{\Pi_2} \prod_{i=1}^{M} \hat{j}_{\Pi_2(i) - \Pi_1(i)},
\end{aligned}
\tag{18}
$$

where Π_i are permutations of the length M and

$$
\hat{j}_n = \int \frac{\mathrm{d}t}{2\pi} j(t) \, \mathrm{e}^{-int}
\tag{19}
$$

are the Fourier coefficients of the source. That is, $I[j]$ is a sum over the Toeplitz determinants of $\hat{j}$,

$$
I[j] = \sum_{M=0}^{\infty} \det T^{(M)}[j],
\tag{20}
$$

where $T^{(M)}$ is an $M \times M$ matrix with the entries

$$
(T^{(M)}[j])_{kl} = \hat{j}_{k-l}.
\tag{21}
$$

One might expect that formula (17) can be generalized to particles with higher charges by taking higher functional derivatives at a single point, or letting some of the t_i's in (17) coincide. However, as above in section 3 this leads to a problem with the infinite self-energy. Luckily, as above in (12), the self-energy can be removed by taking suitable derivatives w.r.t. t_i. For example, the weight for a state with N unit charges and one double charge can be expressed as

$$
\begin{aligned}
\rho_2(t_1, \ldots, t_N, \tau) &= \rho_{\mathrm{D}}(t_1, \ldots, t_N, \tau, \tau + \epsilon) \, \epsilon^{-2} + \mathcal{O}(\epsilon) \\
&= \frac{1}{2!} \left(\frac{\partial}{\partial \tau_2} \right)^2 \rho_{\mathrm{D}}(t_1, \ldots, t_N, \tau_1, \tau_2) \bigg|_{\tau = \tau_1 = \tau_2} \\
&= \frac{1}{2!} \left(\frac{\partial}{\partial \tau_2} \right)^2 \frac{\delta^{N+2} I[j]}{\delta j(t_1) \cdots \delta j(t_N) \delta j(\tau_1) \delta j(\tau_2)} \bigg|_{\substack{j=0 \\ \tau = \tau_1 = \tau_2}},
\end{aligned}
\tag{22}
$$

where $\epsilon^{-2} = \exp(2E_{\mathrm{self}})$ was added to remove the self-energy. As seen from (22) combining the differentiation idea of the previous section (12) to the functional derivative result (17) leads to a 'normalized' second-order functional derivative at a single point,

$$
D_2 \sim \mathrm{e}^{+2E_{\mathrm{self}}} \int \frac{\mathrm{d}t}{2\pi} \frac{\delta^2}{\delta^2 j(t)} = \int \frac{\mathrm{d}\tau}{2\pi} \left[\frac{1}{2!} \left(\frac{\partial}{\partial \tau_2} \right)^2 \frac{\delta^2}{\delta j(\tau_1) \delta j(\tau_2)} \right]_{\tau = \tau_1 = \tau_2},
\tag{23}
$$

where the infinite self-energy has been removed.

We shall now work out a generalization of (23) for D_n with $n > 2$. It can be conveniently expressed in terms of the Fourier modes $\hat{j}_n$ of the source $j(t)$ (see the definition (28) below). Let us consider the case where one particle (at $\mathrm{e}^{i\tau}$) carries an arbitrary integer charge n, which can be added by using the result (A.6) in the appendix A. For $n > 2$ (22) generalizes to

$$
\begin{aligned}
\rho_n(t_1, \ldots, t_N, \tau) &= \prod_{k=2}^{n} \frac{1}{(2k-2)!} \left(\frac{\partial}{\partial \tau_k} \right)^{2k-2} \rho_{\mathrm{D}}(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n) \bigg|_{\tau = \tau_1 = \cdots = \tau_n} \\
&= \frac{(-1)^{n(n-1)/2}}{n! \, [(n-1)!]^n} \prod_{k=1}^{n} \left(\frac{\partial}{\partial \tau_k} \right)^{n-1} \rho_{\mathrm{D}}(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n) \bigg|_{\tau = \tau_1 = \cdots = \tau_n}.
\end{aligned}
\tag{24}
$$

In particular, the definition in terms of derivatives can be written in various forms. Any combination is fine as long as the sum of the orders of the derivatives is $n(n-1)$ and the normalization factor K of the appendix A can be worked out. We shall use the latter form of (24). Then the partition function reads

$$
Z_{C,n}(N) = Z_C(N_1 = N, N_2 = 0, \ldots, N_{n-1} = 0, N_n = 1)
$$

$$
= \frac{(-1)^{n(n-1)/2}}{N!\,n!\,[(n-1)!]^n} \int \prod_{i=1}^{N} \frac{\mathrm{d}t_i}{2\pi} \frac{\mathrm{d}\tau}{2\pi}
$$

$$
\times \left\{ \left(\frac{\partial}{\partial \tau_1}\right)^{n-1} \cdots \left(\frac{\partial}{\partial \tau_n}\right)^{n-1} \frac{\delta^{N+n} I[j]}{\delta j(t_1) \cdots \delta j(t_N) \delta j(\tau_1) \cdots \delta j(\tau_n)} \right\}_{\substack{j=0 \\ \tau=\tau_1=\cdots=\tau_n}}. \quad (25)
$$

The partition function can now be evaluated by using the result (20). The chain rule implies

$$
\frac{\delta}{\delta j(t)} = \sum_{m=-\infty}^{\infty} \frac{\delta \hat{j}_m}{\delta j(t)} \frac{\partial}{\partial \hat{j}_m} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-imt} \frac{\partial}{\partial \hat{j}_m}. \quad (26)
$$

Only the terms that include products of exactly $N+n$ of $\hat{j}_m$'s give nonzero contribution which fixes $M = N + n$ in (20). Hence

$$
Z_{C,n} = \frac{(-1)^{n(n-1)/2}}{N!\,n!\,[(n-1)!]^n} \int \frac{\mathrm{d}\tau}{2\pi} \left\{ \left(\frac{\partial}{\partial \tau_1}\right)^{n-1} \cdots \left(\frac{\partial}{\partial \tau_n}\right)^{n-1} \right. \quad (27)
$$

$$
\left. \times \sum_{m_1=-M}^{M} \cdots \sum_{m_n=-M}^{M} \exp\left[-i\sum_{k=1}^{n} m_k \tau_k\right] \left(\frac{\partial}{\partial \hat{j}_0}\right)^N \prod_{k=1}^{n} \frac{\partial}{\partial \hat{j}_{m_k}} T^{(N+n)}[j] \right\}_{\substack{j=0 \\ \tau=\tau_1=\cdots=\tau_n}}
$$

$$
= \frac{1}{N!} \left(\frac{\partial}{\partial \hat{j}_0}\right)^N \sum_{m_1=-M}^{M} \cdots \sum_{m_n=-M}^{M} \frac{\delta\left(\sum_{k=1}^{n} m_k, 0\right)}{n!\,[(n-1)!]^n} \left[\prod_{k=1}^{n} (m_k)^{n-1} \frac{\partial}{\partial \hat{j}_{m_k}}\right] \det T^{(M)}[j] \Big|_{j=0}, \quad (27)
$$

where $\delta(m,n)$ denotes the Kronecker delta. Here the operator[9]

$$
D_n = \sum_{m_1=-M}^{M} \cdots \sum_{m_n=-M}^{M} \frac{\delta\left(\sum_{k=1}^{n} m_k, 0\right)}{n!\,[(n-1)!]^n} \prod_{k=1}^{n} (m_k)^{n-1} \frac{\partial}{\partial \hat{j}_{m_k}} \quad (28)
$$

generates the particle with the charge n in the gas. The result (28) is valid for $\beta = 2$, but depends on temperature, in general. The temperature dependence arises from the self-energies which can be removed by differentiation similarly as above at all integer-valued inverse temperatures. Thus (28) can be easily generalized to all $\beta = 1, 2, \ldots$. However, the generating functional can be written in terms of Toeplitz determinants as in (18) only for $\beta = 2$.

Generalization of (27) to any number of multiply-charged particles is straightforward. If we set $M = \sum_{n=1}^{n_{\max}} nN_n$ in the definition (28), then[10]

$$
Z_C(N_1, \ldots, N_{n_{\max}}) = \left[\prod_{n=1}^{n_{\max}} \frac{(D_n)^{N_n}}{N_n!}\right] \det T^{(M)}[j] \Big|_{j=0}. \quad (29)
$$

Inserting the definition (21) one can express Z_C as a finite sum similarly as in appendix A (see also the concrete example below).

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[9] Note that D_n are not unique. For example, using the first line in (24) would lead to $D_n = \sum_{m_1,\ldots,m_n=-M}^{M} \delta_{\sum_{k=1}^{n} m_k, 0} \prod_{k=1}^{n} \frac{(-m_k^2)^{k-1}}{(2k-2)!} \frac{\partial}{\partial \hat{j}_{m_k}}$.

[10] Note that $D_1 = \frac{\partial}{\partial \hat{j}_0}$.

Also, a compact formal expression for the grand canonical partition function (6) at $\beta = 2$ immediately follows,

$$Z_{\mathrm{G}}(\hat{z}_1, \ldots, \hat{z}_{n_{\max}}) = \exp\left[\sum_{n=1}^{n_{\max}} \hat{z}_n D_n\right] \sum_{M=0}^{\infty} \det T^{(M)}[j]\Bigg|_{j=0}, \tag{30}$$

where the summation limits in the definition (28) need to be extended from $\pm M$ to $\pm\infty$.

5. An example

As a concrete example of the use of (29) continue from (27) and show how a purely combinatorial result is derived in the case of only one multiply-charged particle. By inserting the explicit form of $\det T^{(M)}[j]$ from (18) we find

$$\begin{aligned}
Z_{\mathrm{C},n} &= \frac{1}{N!} D_1^N D_n \det T^{(M)}[j]\Bigg|_{j=0} \\
&= \frac{1}{N!\,n!\,[(n-1)!]^n} \sum_{m_1=-M}^{M} \cdots \sum_{m_n=-M}^{M} \delta\left(\sum_{k=1}^{n} m_k, 0\right) \prod_{k=1}^{n} (m_k)^{n-1} \\
&\quad \times \sum_{\Pi_1, \Pi_2} (-1)^{\Pi_1}(-1)^{\Pi_2} \prod_{i=1}^{N} \delta_{\Pi_1(i),\Pi_2(i)} \prod_{j=1}^{n} \delta_{\Pi_1(j+n)+m_j,\Pi_2(j+n)}.
\end{aligned} \tag{31}$$

In this special case of only one multiply-charged particle, the condition $\sum_{k=1}^{n} m_k = 0$ follows the others in (31), and may be dropped. The conditions $\Pi_1(i) = \Pi_2(i)$ for $i = 1, \ldots, N$ define a certain 'subdeterminant', which may be written as

$$Z_{\mathrm{C},n} = \frac{1}{[(n-1)!]^n} \sum_{1 \leqslant \ell_1 < \cdots < \ell_n}^{M} \sum_{\tilde{\Pi}} (-1)^{\tilde{\Pi}} \prod_{k=1}^{n} [\ell_k - \ell_{\tilde{\Pi}(k)}]^{n-1}, \tag{32}$$

where $\tilde{\Pi}$ is a permutation of length n.

Acknowledgments

Appendix A. A general combinatorical formula for Z_{C}

In this appendix we derive a general expression for Z_{C} using confluent Vandermonde determinants and compute the consequent combinatorical result. For the general multi-component Coulomb gas, (8) reads

$$\prod_{\mathrm{pairs}} \left| z_i^{(n)} - z_j^{(m)} \right|^{2nm} = |\det A|^2, \tag{A.1}$$

where A is a $M \times M$ confluent Vandermonde matrix with $M = \sum_{n=1}^{n_{\max}} n N_n$. A is defined by

$$A = \left(A^{(1)}\left(z_1^{(1)}\right) \cdots A^{(1)}\left(z_{N_1}^{(1)}\right) \cdots A^{(n_{\max})}\left(z_1^{(n_{\max})}\right) \cdots A^{(n_{\max})}\left(z_{N_{n_{\max}}}^{(n_{\max})}\right) \right), \qquad \text{(A.2)}$$

where each $M \times n$ subblock $A^{(n)}$ corresponds to one particle of the gas and has the entries

$$(A^{(n)}(z))_{ij} = \frac{1}{(j-1)!} \left(\frac{\partial}{\partial z} \right)^{j-1} z^{i-1} = \begin{cases} \binom{i-1}{j-1} z^{i-j} & \text{if} \quad i \geqslant j \\ 0 & \text{if} \quad i < j. \end{cases} \qquad \text{(A.3)}$$

Equation (A.1) with the definition (A.2), (A.3) for A is a known result, but for use elsewhere we will sketch how it follows from the standard Vandermonde determinant (8). Let $f(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n)$ be a function such that

$$f(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n) = g(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n) \prod_{1 \leqslant i < j}^{n} (\tau_i - \tau_j)^{k_{ij}}, \qquad \text{(A.4)}$$

where k_{ij} are positive integers and g stays finite and nonzero as any $\tau_i \to \tau_j$. Then take the m_i th derivative w.r.t. τ_i for each $i = 1, 2, \ldots, n$ of the both sides of (A.4) such that $m_i \geqslant 0$ and

$$\sum_{i=1}^{n} m_i = \sum_{1 \leqslant i < j}^{n} k_{ij}, \qquad \text{(A.5)}$$

and let $\tau = \tau_1 = \cdots = \tau_n$ in the end. On the right-hand side only such terms survive, where all the derivatives act on the product. Thus

$$g(t_1, \ldots, t_N, \tau, \ldots, \tau) = \frac{1}{K} \left(\frac{\partial}{\partial \tau_1} \right)^{m_1} \cdots \left(\frac{\partial}{\partial \tau_n} \right)^{m_n} f(t_1, \ldots, t_N, \tau_1, \ldots, \tau_n) \Bigg|_{\tau = \tau_1 = \cdots = \tau_n}, \qquad \text{(A.6)}$$

where K is the constant integer[11]

$$K = \left(\frac{\partial}{\partial \tau_1} \right)^{m_1} \cdots \left(\frac{\partial}{\partial \tau_n} \right)^{m_n} \prod_{1 \leqslant i < j}^{n} (\tau_i - \tau_j)^{k_{ij}} = \tilde{K} \prod_{k=1}^{n} m_k!, \qquad \text{(A.7)}$$

where $\tilde{K}$ is the coefficient of the monomial $\tau_1^{m_1} \cdots \tau_n^{m_n}$ in $\prod_{1 \leqslant i < j}^{n} (\tau_i - \tau_j)^{k_{ij}}$. To prove (A.1) for the case of N unit charges and one particle with charge n, we apply the result (A.6) to

$$f(z_1, \ldots, z_N, w_1, \ldots, w_n) = \Delta(z_1, \ldots, z_N, w_1, \ldots, w_n)$$

$$= \prod_{1 \leqslant i < j}^{n} (w_i - w_j) \times \prod_{1 \leqslant i < j}^{N} (z_i - z_j) \prod_{i=1}^{N} \prod_{j=1}^{n} (z_i - w_j)$$

$$= \prod_{1 \leqslant i < j}^{n} (w_i - w_j) \times g(z_1, \ldots, z_N, w_1, \ldots, w_n) \qquad \text{(A.8)}$$

and take $m_k = k - 1$. After calculating the integer $K = \prod_{k=1}^{n} (k-1)!$ we find

$$\prod_{1 \leqslant i < j}^{N} (z_i - z_j) \prod_{i=1}^{N} (z_i - w)^n = g(z_1, \ldots, z_N, w, \ldots, w)$$

$$= \prod_{k=1}^{n} \frac{1}{(k-1)!} \left(\frac{\partial}{\partial w_k} \right)^{k-1} \Delta(z_1, \ldots, z_N, w_1, \ldots, w_n) \Bigg|_{w = w_1 = \cdots = w_n}$$

$$= \pm \det(A^{(1)}(z_1) \cdots A^{(1)}(z_N) A^{(n)}(w)), \qquad \text{(A.9)}$$

---

[11] The result only makes sense if $K \neq 0$.

where we used the determinant form of Δ in (8) and (A.3) in the last line to complete the proof. The general result (equation (A.1) with (A.2)) follows by a straightforward induction.

In the rest of this appendix, we use the explicit form (A.2), (A.3) of A to find a combinatorical formula for Z_C, the constant term in all $z_i^{(n)}$ of (A.1) when expanded into an analytic function of $z_i^{(n)}$. Note that since we only need the determinant of A and $\left| z_i^{(n)} \right| = 1$ we may multiply any column by an arbitrary power of $z_i^{(n)}$. Using in addition elementary column operations $A^{(n)}(z)$ may be replaced by

$$(\tilde{A}^{(n)}(z))_{ij} = \frac{i^{j-1}}{(j-1)!} z^i. \tag{A.10}$$

Then we find

$$|\det A|^2 = \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_M} \prod_{n=1}^{n_{\max}} \prod_{k=1}^{N_n} \left[\prod_{s=1}^{n} \frac{(i_{\ell(n,k)+s})^{s-1}}{(s-1)!} \left(z_k^{(n)} \right)^{i_{\ell(n,k)+s}} \right] \right|^2$$

$$= \left| \sum_{\{i\}} \varepsilon_{i_1 \cdots i_M} \prod_{n=1}^{n_{\max}} \prod_{k=1}^{N_n} \frac{1}{n!(n-1)!\cdots 1!} \Delta(i_{\ell(n,k)+1}, \ldots, i_{\ell(n,k)+n}) \left(z_k^{(n)} \right)^{\sum_{s=1}^{n} i_{\ell(n,k)+s}} \right|^2,$$

where the Vandermonde matrices of the permutation variables in the last form are obtained after antisymmetrization. The function

$$\ell(n,k) = \sum_{m=1}^{n-1} m N_m + (k-1)n \tag{A.11}$$

is only needed for picking up the permutation variable i with the correct index.

The constant term of $|\det A|^2 = \det A^* \det A$ is

$$Z_C(N_1, \ldots, N_{n_{\max}}) \prod_{n=1}^{n_{\max}} N_n! = \sum_{\{i\},\{j\}} \varepsilon_{i_1 \cdots i_M} \varepsilon_{j_1 \cdots j_M} \prod_{n=1}^{n_{\max}} \prod_{k=1}^{N_n} \frac{1}{[n!(n-1)!\cdots 1!]^2}$$

$$\times \Delta(i_{\ell(n,k)+1}, \ldots, i_{\ell(n,k)+n}) \Delta(j_{\ell(n,k)+1}, \ldots, j_{\ell(n,k)+n})$$

$$\times \delta(i_{\ell(n,k)+1} + \cdots + i_{\ell(n,k)+n}, j_{\ell(n,k)+1} + \cdots + j_{\ell(n,k)+n}), \tag{A.12}$$

where $\delta(i,j) = \delta_{ij}$ is the Kronecker δ-symbol. For $n = 1$ the δ restrictions give simply $i_k = j_k$. Using these the result evaluates to

$$Z_C \prod_{n=2}^{n_{\max}} N_n! = \sum_{S,\{i\},\{j\}} \varepsilon_{i_1 \cdots i_K} \varepsilon_{j_1 \cdots j_K} \prod_{n=2}^{n_{\max}} \prod_{k=1}^{N_n} \frac{1}{[n!(n-1)!\cdots 1!]^2}$$

$$\times \Delta(S(i_{\ell'(n,k)+1}), \ldots, S(i_{\ell'(n,k)+n})) \Delta(S(j_{\ell'(n,k)+1}), \ldots, S(j_{\ell'(n,k)+n}))$$

$$\times \delta \left(\sum_{s=1}^{n} S(i_{\ell'(n,k)+s}), \sum_{s=1}^{n} S(j_{\ell'(n,k)+s}) \right), \tag{A.13}$$

where $K = M - N_1$, the first sum goes over all increasing functions $S : \{1, \ldots, K\} \to \{1, \ldots, M\}$ (so that $i < j \Leftrightarrow S(i) < S(j)$), and $\ell'(n,k) = \ell(n,k) - N_1$.

Due to symmetry, one can add the restrictions $i_{\ell'(n,k)+1} < i_{\ell'(n,k+1)+1}$ (for all $n > 1$ and $1 \leqslant k < N_n$), and $i_{\ell'(n,k)+s} < i_{\ell'(n,k)+s+1}$, $j_{\ell'(n,k)+s} < j_{\ell'(n,k)+s+1}$ (for all $n > 1, k$, and $1 \leqslant s < n$) and multiply by the ratio of numbers of terms $\left(\prod_{n=2}^{n_{\max}} n!^{N_n} N_n! \right)$. Then the result

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becomes

$$
\begin{aligned}
Z_C = \sideset{}{'}\sum_{S,\{i\},\{j\}} \varepsilon_{i_1 \cdots i_K} \varepsilon_{j_1 \cdots j_K} \prod_{n=2}^{n_{\max}} \prod_{k=1}^{N_n} \frac{1}{[(n-1)! \cdots 1!]^2} \\
\times\ \Delta(S(i_{\ell'(n,k)+1}), \ldots, S(i_{\ell'(n,k)+n})) \Delta(S(j_{\ell'(n,k)+1}), \ldots, S(j_{\ell'(n,k)+n})) \\
\times\ \delta\left(\sum_{s=1}^{n} S(i_{\ell'(n,k)+s}), \sum_{s=1}^{n} S(j_{\ell'(n,k)+s}) \right),
\end{aligned}
\tag{A.14}
$$

where the prime indicates the presence of the above restrictions.

Appendix B. Relation to field theory

In this appendix we present a compact formulation (for all inverse temperatures β) of the generating functional (16) using the language of field theory. We use the operator $X(t)$ that lies at $z = e^{it}$ on the boundary of the unit disk. Its self contraction

$$
\langle X(t_1) X(t_2) \rangle = \log |e^{it_1} - e^{it_2}|^2
\tag{B.1}
$$

is essentially the Coulomb potential between charges at e^{it_1} and at e^{it_2}. Then the generating functional (16) reads

$$
I[j] = \left\langle \exp\left[\int \frac{dt}{2\pi} j(t) : e^{\sqrt{\beta/2} X(t)} : \right] \right\rangle
\tag{B.2}
$$

as can be verified by using the Wick theorem. The canonical partition function of the (single-component) Dyson gas reads

$$
Z_{C,D}(N) = \frac{1}{N!} \left\langle \left[\int \frac{dt}{2\pi} : \exp(\sqrt{\beta/2} X(t)) : \right]^N \right\rangle = \frac{1}{N!} \left[\int \frac{dt}{2\pi} \frac{\delta}{\delta j(t)} \right]^N I[j] \Bigg|_{j=0}
\tag{B.3}
$$

where the last step can be checked using (B.2).

In the case of the multi-component gas

$$
Z_C(N_1, \ldots, N_{n_{\max}}) = \frac{1}{\prod_{n=1}^{n_{\max}} (N_n)!} \left\langle \prod_{n=1}^{n_{\max}} \left[\int \frac{dt}{2\pi} : e^{n\sqrt{\beta/2} X(t)} : \right]^{N_n} \right\rangle
\tag{B.4}
$$

so that one would naively expect that

$$
Z_C(N_1, \ldots, N_{n_{\max}}) \sim \frac{1}{\prod_{n=1}^{n_{\max}} (N_n)!} \prod_{n=1}^{n_{\max}} \left[\int \frac{dt}{2\pi} \frac{\delta^n}{(\delta j(t))^n} \right]^{N_n} I[j] \Bigg|_{j=0}.
\tag{B.5}
$$

However, (B.5) fails in general since the higher order functional derivatives at the same point are not compatible with the normal ordering: the right-hand side includes self contractions between fields at the same point. These are the self-energies of the multiply-charged particles which were discussed in the text. Equation (B.5) holds only after the subtraction of the self-energies, which was done for $\beta = 2$ explicitly in section 4, leading to (29). Note also that the grand canonical partition function can be written as (see [12])

$$
Z_G(\hat{z}_1, \ldots, \hat{z}_{n_{\max}}) = \left\langle \exp\left[\sum_{n=1}^{n_{\max}} \hat{z}_n \int \frac{dt}{2\pi} : e^{n\sqrt{\beta/2} X(t)} : \right] \right\rangle
\tag{B.6}
$$

which explains the exponential form of the result (30).

References

[1] Saleur H Lectures on non perturbative field theory and quantum impurity problems 1998 *Preprint* cond-mat/9812110

[2] Forrester P J 1988 Solvable isotherms for a two-component system of charged rods on a line *J. Stat. Phys.* **51** 457
   Forrester P J 1989 Exact results for correlations in a two-component log-gas *J. Stat. Phys.* **59** 57

[3] Fendley P, Lesage F and Saleur H 1995 Solving 1-d plasmas and 2-d boundary problems using Jack polynomials and functional relations *J. Stat. Phys.* **79** 799 (*Preprint* hep-th/9409176)

[4] Samaj L and Travenec I 2000 Thermodynamic properties of the two-dimensional two-component plasma *J. Stat. Phys.* **101** 713 (*Preprint* cond-mat/0004021)

[5] Samaj L 2001 Thermodynamic properties of the one-dimensional two-component log-gas *J. Stat. Phys.* **105** 175 (*Preprint* cond-mat/0102332)

[6] Forrester P J 1984 Interpretation of an exactly solvable two-component plasma *J. Stat. Phys.* **35** 77

[7] Dyson F J 1962 *J. Math. Phys.* **3** 140

[8] Mehta M L 1991 *Random Matrices* 2nd edn (New York:: Academic)

[9] Wilson K G 1962 *J. Math. Phys.* **3** 1040
   Good I J 1970 *J. Math. Phys.* **11** 1884

[10] Jokela N, Keski-Vakkuri E and Majumder J 2008 *Phys. Rev.* D **77** 023523 (*Preprint* 0709.1318)

[11] Jokela N, Järvinen M and Keski-Vakkuri E in preparation

[12] Jokela N, Järvinen M, Keski-Vakkuri E and Majumder J 2008 *J. Phys. A: Math. Theor.* **41** 015402 (*Preprint* 0705.1916)

[13] Kalman D 1984 *Math. Mag.* **57** 15

[14] Balasubramanian V, Keski-Vakkuri E, Kraus P and Naqvi A 2005 *Commun. Math. Phys.* **257** 363 (*Preprint* hep-th/0404039)
   Jokela N, Keski-Vakkuri E and Majumder J 2006 *Phys. Rev.* D **73** 046007 (*Preprint* hep-th/0510205)